J. Math. Anal. Appl. 381 (2011) 521–529

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# Boundedness in a chemotaxis model with oxygen consumption by bacteria ☆

Youshan Tao

*Department of Applied Mathematics, Dong Hua University, Shanghai 200051, PR China*

ARTICLE INFO

*Article history:*
Received 12 December 2010
Available online 16 February 2011
Submitted by Goong Chen

*Keywords:*
Chemotaxis
Global existence
Boundedness

ABSTRACT

This paper deals with the Keller–Segel model

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \end{cases}$$

under homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, with nonnegative initial data $(u(\cdot,0), v(\cdot,0)) \in (W^{1,r}(\Omega))^2$ (for some $r > n$), $\|u(\cdot,0)\|_{L^1(\Omega)} > 0$ and $\|v(\cdot,0)\|_{L^\infty(\Omega)} > 0$. This model describes bacteria movement toward the concentration gradient of the oxygen that is consumed by the bacteria. It is proved that if

$$0 < \chi \leqslant \frac{1}{6(n+1)\|v(\cdot,0)\|_{L^\infty(\Omega)}}$$

then the corresponding initial–boundary value problem possesses a unique global solution that is uniformly bounded.



## 1. Introduction

This paper is concerned with the global existence and the boundedness of solutions to the chemotaxis system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \\ \partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

for the unknown $u = u(x,t)$ and $v = v(x,t)$ which denote bacteria density and oxygen concentration, respectively. Here $\chi > 0$ is a parameter referred to as chemosensitivity, $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ is a bounded domain with smooth boundary, $u_0$ and $v_0$ are given nonnegative functions, and $\partial_\nu$ denotes the differentiation with respect to the outward normal derivative on $\partial\Omega$.

In 1971, Keller and Segel [15] proposed the following well-known one-dimensional Keller–Segel model

$$\begin{cases} u_t = u_{xx} - \chi \left(u v^{-1} v_x\right)_x, \\ v_t = \varepsilon v_{xx} - u f(v), \end{cases} \tag{1.2}$$

☆ This work was supported by the National Natural Science Foundation of China.
*E-mail address:* taoys@dhu.edu.cn.

0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2011.02.041

which describes the traveling band behavior of bacteria due to the chemotactic response (i.e., the biased movement of bacteria to the oxygen concentration gradient) observed in experiment [1]. In model (1.2), $\varepsilon > 0$ denotes the diffusion coefficient of the oxygen, and $f(v)$ is a kinetic function describing the chemical reaction between bacteria and the oxygen. When $\varepsilon = 0$ and $f(v) = \alpha > 0$, the existence of traveling wave solutions of (1.2) was established by Keller and Segel themselves [15]. When $\varepsilon > 0$ and $f(v) = \alpha > 0$, the existence and linear instability of traveling solutions of (1.2) were proved by Nagai and Ikeda [20]. When $\varepsilon > 0$ and $f(v) = \alpha v$ $(\alpha > 0)$, Li and Wang [18] recently established the existence and the nonlinear stability of traveling wave solutions to a system of conservation laws which is transformed from model (1.2) by a change of variable via $w = -v^{-1}v_x = -(\ln v)_x$, which was initially introduced by Wang and Hillen in [27]. Model (1.1) can be regarded as the higher-dimensional version of model (1.2) with $f(v) = v$ and signal-independent sensitivity.

To describe the motion of oxygen-driven swimming bacteria in an incompressible fluid, Tuval et al. [26] proposed the following coupled Keller–Segel–Navier–Stokes model

$$\begin{cases} u_t + V\cdot\nabla u = \Delta u - \nabla\cdot\big(u\chi(v)\nabla v\big),\\ v_t + V\cdot\nabla v = \Delta v - uf(v),\\ V_t + V\cdot\nabla V + \nabla p_e - \eta\Delta V + u\nabla\phi = 0,\\ \nabla\cdot V = 0, \end{cases} \tag{1.3}$$

where, as before, $u$ and $v$ denote the bacterium density and the oxygen concentration, respectively, and $V$ represents the velocity field of the fluid subject to an incompressible Navier–Stokes equation with pressure $p_e$ and viscosity $\eta$ and a gravitational force $\nabla\phi$. The function $\chi(v)$ measures the chemotactic sensitivity, $f(v)$ is the consumption rate of the oxygen by the bacteria, and $\phi$ is a given potential function. In model (1.3), bacteria and the oxygen are transported with the fluid. Lorz [19] proved local existence of solutions to (1.3), whereas the authors in [9] proved global existence of classical solutions near constant states in three space dimensions. When the nonlinear convective term $V\cdot\nabla V$ is ignored (i.e., $V\cdot\nabla V \equiv 0$) in the third equation of (1.3), the authors in [9] proved global existence of certain weak solutions to the corresponding Keller–Segel–Stokes model in two space dimensions under suitable smallness assumptions on either $\phi$ or $v(\cdot,0)$. When the nonlinear convective term $V\cdot\nabla V$ is ignored in the third equation of (1.3) and the diffusion term $\Delta u$ in the first equation of (1.3) is replaced by a porous medium-type diffusion term $\Delta u^m$, Tao and Winkler [25] recently proved that global bounded weak solutions exist whenever $m > 1$ and initial data $(u_0, v_0, V_0)$ are sufficiently regular satisfying $u_0 \geqslant 0$ and $v_0 \geqslant 0$. This extends a previous result by Di Francesco, Lorz and Markowich [8] which asserts global existence of weak solutions under the constraint $m \in (\frac{3}{2}, 2]$. If the flow of fluid is ignored (i.e., $V \equiv 0$) or the fluid is stationary, then model (1.3) with $f(v) = v$ and $\chi(v) = \text{const.} := \chi$ yields the fundamental chemotaxis model (1.1). To better understand model (1.3), it is necessary to first study model (1.1), which is the focus of the present work.

Before stating our main results on model (1.1), we should mention the following classical Keller–Segel model [16]

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\nabla v), & x\in\Omega,\ t>0,\\ v_t = \Delta v - v + u, & x\in\Omega,\ t>0,\\ \partial_\nu u = \partial_\nu v = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x\in\Omega, \end{cases} \tag{1.4}$$

where $u$ denotes the cell density and $v$ represents the concentration of the chemical signal secreted by cells. Model (1.4) is known to describe the aggregation of the paradigm species *Dictyostelium discoideum* [16], and it is believed to be present in many biologically meaningful chemotactic processes [12]. Model (1.4) has been extensively studied during past three decades. If $n = 1$, then all solutions of (1.4) are global in time and bounded [23]; if $n = 2$ and $\int_\Omega u_0 < 4\pi$, then the solution will be global and bounded [21]; if $n \geqslant 3$ and $\|u_0\|_{L^{n/2+\delta}(\Omega)}$ and $\|\nabla v_0\|_{L^{n+\delta}(\Omega)}$ are small for any $\delta > 0$, then the solution is global and bounded [28]. Roughly speaking, under suitable smallness assumptions on the initial data $u_0$ and $v_0$, the solution to (1.4) is global and bounded in higher dimensions ($n \geqslant 2$). On the other hand, if $n = 2$, then for almost every $M > 4\pi$ there exist smooth initial data $(u_0, v_0)$ with $\int_\Omega u_0 = M$ such that corresponding solution of (1.4) blows up either in finite or infinite time [13] and that the radially symmetric solutions may even blow up in finite time [11]; if $n \geqslant 3$, then for all $M > 0$ there exist initial data with $\int_\Omega u_0 = M$ such that the radially symmetric solution will be unbounded [28].

The main difference between model (1.1) and model (1.4) is that the signal in the former is *consumed* by cells whereas the signal in the latter is *produced* by cells. The present study shows that model (1.1) may possess some properties which are quite different from those of model (1.4). More precisely, our main result reads as follows.

**Theorem 1.1.** *If $n \geqslant 2$, the initial data $u_0$ and $v_0$ are nonnegative, $(u_0, v_0) \in (W^{1,r}(\Omega))^2$ for some $r > n$, $\|u_0\|_{L^1(\Omega)} > 0$, $\|v_0\|_{L^\infty(\Omega)} > 0$ and*

$$0 < \chi \leqslant \frac{1}{6(n+1)\|v_0\|_{L^\infty(\Omega)}}, \tag{1.5}$$

*then* (1.1) *possesses a unique global classical solution that is bounded in $\Omega \times (0,\infty)$.*

As aforementioned, under suitable *smallness* assumptions on the initial data $u_0$ and $v_0$, the solution to model (1.4) is global and bounded in higher dimensions ($n \geqslant 2$); on the other hand, it is well known that there exist blow-up solutions to model (1.4) in higher dimensions for large initial data $u_0$. However, Theorem 1.1 shows that the global existence or blow-up of solutions to model (1.1) is independent of the initial data $u_0$. Unfortunately we have to leave open here the question that whether there exists a blow-up solution to model (1.1) in higher dimensions for large initial data $v_0$ or large chemotactic parameter $\chi$ such that (1.5) does not hold.

A key step of the proof of our main results is to establish a bound for $u(\cdot,t)$ in $L^{n+1}(\Omega)$. To this end, we need to estimate $\int_\Omega u^{n+1}\varphi(v)$ with some weight function $\varphi(v)$ which is uniformly bounded both from above and below by positive constants. This approach was developed by Winkler in [29] for studying a chemotaxis system which can be regarded as an extension of model (1.4) with signal-dependent sensitivity. The choice of the weight function is crucial and technical for the proof of the results in [29]. However, the weight function $\varphi(v)$ chosen in the present paper greatly differs from that in [29] due to that model (1.1) and the model studied in [29] are different as explained before.

## 2. Local existence

The following statement concerning local existence of classical solution can be proved by well-established methods involving standard parabolic regularity theory [3] and an appropriate fixed point framework (for details see [14,29,30] or [5], for instance).

**Lemma 2.1.** *Let $u_0$ and $v_0$ be nonnegative and satisfy $(u_0,v_0) \in (W^{1,r}(\Omega))^2$ for some $r > n$. Then problem* (1.1) *has a unique local in time classical solution*

$$(u,v) \in \big(C\big([0,T_{max}); W^{1,r}(\Omega)\big) \cap C^{2,1}\big(\bar{\Omega} \times (0,T_{max})\big)\big)^2,$$

*where $T_{max}$ denotes the maximal existence time. Moreover, $u$ and $v$ satisfy the inequalities*

$$u \geqslant 0 \quad \textit{and} \quad 0 \leqslant v \leqslant \|v_0\|_{L^\infty(\Omega)} \quad \textit{in } \Omega \times (0,T_{max}); \tag{2.1}$$

*if for each $T > 0$ there exists a constant $C(T)$ (depending on $T$ and $\|(u_0,v_0)\|_{W^{1,r}(\Omega)}$ only) such that*

$$\big\|\big(u(t),v(t)\big)\big\|_{L^\infty(\Omega)} \leqslant C(T), \quad 0 < t < \min\{T, T_{max}\}, \tag{2.2}$$

*then $T_{max} = +\infty$; and the total mass of $u$ evolves according to the identity*

$$\big\|u(t)\big\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \quad \textit{for all } t \in (0,T_{max}). \tag{2.3}$$

**Proof.** As done in [4,7], let $\omega = (u,v) \in \mathbb{R}^2$. Then the system (1.1) can be reformulated as the following triangular system:

$$\begin{cases} \omega_t = \nabla \cdot \big(A(\omega)\nabla\omega\big) + \mathcal{F}(\omega), \\ \dfrac{\partial\omega}{\partial\nu} = 0 \quad \text{on } \partial\Omega \times [0,+\infty), \\ \omega(\cdot,0) = (u_0,v_0) \quad \text{in } \Omega, \end{cases} \tag{2.4}$$

where

$$A(\omega) = \begin{pmatrix} 1 & -\chi u \\ 0 & 1 \end{pmatrix}, \qquad \mathcal{F}(\omega) = \begin{pmatrix} 0 \\ -uv \end{pmatrix}.$$

Then, Theorems 14.4 and 14.6 of [3] are applicable. The first one says that there exists a unique maximal weak $W^{1,r}$-solution. The second one asserts that the solution is a classical solution and the equation is verified point-wise.

Furthermore, if (2.2) holds, then we can invoke Theorem 15.5 of [3] to conclude that $T_{max} = \infty$.

Finally, (2.1) follows from the maximum principle [17], whereas (2.3) immediately result upon integration. □

Before closing this section, we note that if we prescribe more stringent conditions on the initial data, such as $(u_0,v_0) \in C^{2+\gamma_0}(\bar{\Omega})$ with $0 < \gamma_0 < 1$, then one can directly and easily establish the local existence in the function space $C^{2+\gamma_0,(2+\gamma_0)/2}(\Omega \times [0,T_0])$ for some small $T_0 > 0$ by a fixed point argument without employing the abstract Amann's theory (cf. [24], for instance).

## 3. Proof of the main results

The main step towards global existence and boundedness of solutions is to establish uniform bound of the bacteria density $u(\cdot,t)$ in the space $L^{n+1}(\Omega)$. This is accomplished by estimating some associated weighted integral $\int_\Omega u^{n+1}\varphi(v)$ with a weight function $\varphi(v)$ which is uniformly bounded from above and below by positive constants.

**Lemma 3.1.** *Let the assumptions in Theorem* 1.1 *hold. Then there exists a constant* $C > 0$ *such that the first component of solution of* (1.1) *satisfies*

$$\|u(\cdot,t)\|_{L^{n+1}(\Omega)} \leqslant C \quad \text{for all } t \in (0, T_{max}). \tag{3.1}$$

**Proof.** Set

$$k := n+1, \qquad \beta := \sqrt{\frac{n}{24(n+1)}} \cdot \frac{1}{\|v_0\|_{L^\infty(\Omega)}}$$

and define

$$\varphi(s) := e^{(\beta s)^2} \quad \text{for all } 0 \leqslant s \leqslant \|v_0\|_{L^\infty(\Omega)}.$$

By direct calculation we obtain from (1.1) that

$$\begin{aligned}
\frac{1}{k}\frac{d}{dt}\int_\Omega u^k\varphi(v) &= \int_\Omega u^{k-1}\varphi(v)u_t + \frac{1}{k}\int_\Omega u^k\varphi'(v)v_t \\
&= \int_\Omega u^{k-1}\varphi(v)\Delta u - \int_\Omega u^{k-1}\varphi(v)\chi\nabla\cdot(u\nabla v) + \frac{1}{k}\int_\Omega u^k\varphi'(v)\Delta v - \frac{1}{k}\int_\Omega u^{k+1}v\varphi'(v) \\
&= -(k-1)\int_\Omega u^{k-2}\varphi(v)|\nabla u|^2 - \int_\Omega u^{k-1}\varphi'(v)\nabla u\cdot\nabla v \\
&\quad + \chi(k-1)\int_\Omega u^{k-1}\varphi(v)\nabla u\cdot\nabla v + \chi\int_\Omega u^k\varphi'(v)|\nabla v|^2 \\
&\quad - \int_\Omega u^{k-1}\varphi'(v)\nabla u\cdot\nabla v - \frac{1}{k}\int_\Omega u^k\varphi''(v)|\nabla v|^2 - \frac{1}{k}\int_\Omega u^{k+1}v\varphi'(v).
\end{aligned}$$

Since $v \geqslant 0$ and $\varphi'(s) \geqslant 0$ for all $s \geqslant 0$, we thus have

$$\begin{aligned}
&\frac{1}{k}\frac{d}{dt}\int_\Omega u^k\varphi(v) + (k-1)\int_\Omega u^{k-2}\varphi(v)|\nabla u|^2 + \frac{1}{k}\int_\Omega u^k\varphi''(v)|\nabla v|^2 \\
&\quad \leqslant -2\int_\Omega u^{k-1}\varphi'(v)\nabla u\cdot\nabla v + \chi(k-1)\int_\Omega u^{k-1}\varphi(v)\nabla u\cdot\nabla v + \chi\int_\Omega u^k\varphi'(v)|\nabla v|^2.
\end{aligned} \tag{3.2}$$

By Young's inequality,

$$-2\int_\Omega u^{k-1}\varphi'(v)\nabla u\cdot\nabla v \leqslant \frac{k-1}{4}\int_\Omega u^{k-2}\varphi(v)|\nabla u|^2 + \frac{4}{k-1}\int_\Omega u^k\frac{\varphi'^2(v)}{\varphi(v)}|\nabla v|^2$$

and

$$\chi(k-1)\int_\Omega u^{k-1}\varphi(v)\nabla u\cdot\nabla v \leqslant \frac{k-1}{4}\int_\Omega u^{k-2}\varphi(v)|\nabla u|^2 + \chi^2(k-1)\int_\Omega u^k\varphi(v)|\nabla v|^2.$$

Thus, (3.2) yields

$$\begin{aligned}
&\frac{1}{k}\frac{d}{dt}\int_\Omega u^k\varphi(v) + \frac{k-1}{2}\int_\Omega u^{k-2}\varphi(v)|\nabla u|^2 + \frac{1}{k}\int_\Omega u^k\varphi''(v)|\nabla v|^2 \\
&\quad \leqslant \frac{4}{k-1}\int_\Omega u^k\frac{\varphi'^2(v)}{\varphi(v)}|\nabla v|^2 + \chi^2(k-1)\int_\Omega u^k\varphi(v)|\nabla v|^2 + \chi\int_\Omega u^k\varphi'(v)|\nabla v|^2.
\end{aligned} \tag{3.3}$$

Next we show that the three terms on the right-hand side of (3.3) are dominated by $\frac{1}{k}\int_\Omega u^k\varphi''(v)|\nabla v|^2$. To this end, we first compute

$$I_1(s) := \frac{4}{k-1}\frac{\varphi'^2(s)}{\varphi(s)} = \frac{4}{k-1}\cdot 4\beta^4 s^2 e^{(\beta s)^2},$$
$$I_2(s) := \chi^2(k-1)\varphi(s) = \chi^2(k-1)e^{(\beta s)^2},$$
$$I_3(s) := \chi\varphi'(s) = 2\chi\beta^2 s e^{(\beta s)^2}$$

and

$$I_4(s) := \frac{1}{k}\varphi''(s) = \frac{1}{k}\cdot 2\beta^2 e^{(\beta s)^2} + \frac{1}{k}\cdot 4\beta^4 s^2 e^{(\beta s)^2}$$

for $s \geqslant 0$. Then, using $0 \leqslant v \leqslant \|v_0\|_{L^\infty(\Omega)}$ and assumption (1.5), we estimate

$$\begin{aligned}
\frac{I_1(v)}{\frac{1}{3}I_4(v)} &\leqslant \frac{\frac{4}{k-1}\cdot 4\beta^4 v^2 e^{(\beta v)^2}}{\frac{1}{3}\cdot\frac{1}{k}\cdot 2\beta^2 e^{(\beta v)^2}} \\
&= \frac{24k}{k-1}(\beta v)^2 \\
&\leqslant \frac{24(n+1)}{n}\cdot\left(\beta\|v_0\|_{L^\infty(\Omega)}\right)^2 \\
&\leqslant 1,
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
\frac{I_2(v)}{\frac{1}{3}I_4(v)} &\leqslant \frac{\chi^2(k-1)e^{(\beta v)^2}}{\frac{1}{3}\cdot\frac{1}{k}\cdot 2\beta^2 e^{(\beta v)^2}} \\
&= \frac{3k(k-1)\chi^2}{2\beta^2} \\
&\leqslant \frac{3n(n+1)}{2}\cdot\frac{24(n+1)\|v_0\|^2_{L^\infty(\Omega)}}{n}\cdot\frac{1}{36(n+1)^2\|v_0\|^2_{L^\infty(\Omega)}} \\
&= 1
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
\frac{I_3(v)}{\frac{1}{3}I_4(v)} &\leqslant \frac{2\chi\beta^2 v e^{(\beta v)^2}}{\frac{1}{3}\cdot\frac{1}{k}\cdot 2\beta^2 e^{(\beta v)^2}} \\
&= 3k\chi v \\
&\leqslant 3(n+1)\cdot\frac{1}{6(n+1)\|v_0\|_{L^\infty(\Omega)}}\cdot\|v_0\|_{L^\infty(\Omega)} \\
&= \frac{1}{2} < 1.
\end{aligned} \tag{3.6}$$

From (3.3)–(3.6) we find that

$$\frac{1}{k}\frac{d}{dt}\int_\Omega u^k\varphi(v) + \frac{k-1}{2}\int_\Omega u^{k-2}\varphi(v)|\nabla u|^2 \leqslant 0 \tag{3.7}$$

for all $t \in (0, T_{max})$. Note that for all $0 \leqslant v \leqslant \|v_0\|_{L^\infty(\Omega)}$ we have

$$1 \leqslant \varphi(v) \leqslant e^{(\beta\|v_0\|_{L^\infty(\Omega)})^2} := b > 1. \tag{3.8}$$

Hence,

$$\begin{aligned}
\int_\Omega u^{k-2}\varphi(v)|\nabla u|^2 &\geqslant \int_\Omega u^{k-2}|\nabla u|^2 \\
&= \frac{4}{k^2}\int_\Omega \left|\nabla u^{\frac{k}{2}}\right|^2.
\end{aligned} \tag{3.9}$$

On the other hand, we obtain from (3.8), (2.3) and the Gagliardo–Nirenberg inequality (cf. [10] or [22], for instance) that

$$\begin{aligned}\int_\Omega u^k\varphi(v) &\leqslant b\int_\Omega u^k\\ &= b\|u^{\frac{k}{2}}\|^2_{L^2(\Omega)}\\ &\leqslant b\big(c_1\|\nabla u^{\frac{k}{2}}\|^a_{L^2(\Omega)}\|u^{\frac{k}{2}}\|^{1-a}_{L^{\frac{2}{k}}(\Omega)}+c_2\|u^{\frac{k}{2}}\|_{L^{\frac{2}{k}}(\Omega)}\big)^2\\ &\leqslant 2b\big(c_1^2\|u_0\|^{k(1-a)}_{L^1(\Omega)}\|\nabla u^{\frac{k}{2}}\|^{2a}_{L^2(\Omega)}+c_2^2\|u_0\|^k_{L^1(\Omega)}\big)\end{aligned} \tag{3.10}$$

holds with some positive constants $c_1$, $c_2$ and

$$a=\frac{\frac{k}{2}-\frac{1}{2}}{\frac{k}{2}+\frac{1}{n}-\frac{1}{2}}\in(0,1). \tag{3.11}$$

This, along with Young's inequality, yields

$$\|\nabla u^{\frac{k}{2}}\|^{2a}_{L^2(\Omega)}\leqslant\|\nabla u^{\frac{k}{2}}\|^2_{L^2(\Omega)}+1. \tag{3.12}$$

Combining (3.10) and (3.12) we obtain

$$\int_\Omega u^k\varphi(v)\leqslant c_3\|\nabla u^{\frac{k}{2}}\|^2_{L^2(\Omega)}+c_4, \tag{3.13}$$

where $c_3:=2bc_1^2\|u_0\|^{k(1-a)}_{L^1(\Omega)}>0$ and $c_4:=c_3+2bc_2^2\|u_0\|^k_{L^1(\Omega)}>0$ due to $\|u_0\|_{L^1(\Omega)}>0$. Hence, combining (3.9) and (3.13) entails that

$$\int_\Omega u^{k-2}\varphi(v)|\nabla u|^2\geqslant c_5\int_\Omega u^k\varphi(v)-c_6 \tag{3.14}$$

where $c_5:=\frac{4}{k^2c_3}>0$ and $c_6:=\frac{4c_4}{k^2c_3}>0$. Inserting (3.14) into (3.7) we obtain

$$\frac{1}{k}\frac{d}{dt}\int_\Omega u^k\varphi(v)+\frac{(k-1)c_5}{2}\int_\Omega u^k\varphi(v)\leqslant\frac{(k-1)c_6}{2},$$

which entails that

$$\int_\Omega u^k\varphi(v)\leqslant\frac{c_6}{c_5}+b\int_\Omega u_0^k:=c_7, \tag{3.15}$$

where $c_7$ is independent of $T_{max}$. This, along with (3.8), yields (3.1). $\square$

Before deriving a uniform bound on $u$, let us first collect two well-known facts concerning the Laplacian in $\Omega$ equipped with homogeneous Neumann boundary conditions (cf. [14], for instance). Firstly, the operator $-\Delta+1$ is sectorial in $L^p(\Omega)$ and therefore possesses closed fractional powers $(-\Delta+1)^\theta$, $\theta\in(0,1)$, with dense domain $D((-\Delta+1)^\theta)$. If $p\in[1,\infty]$ and $q\in(1,\infty)$ then with some constant $c>0$, for all $w\in D((-\Delta+1)^\theta)$ we have

$$\|w\|_{W^{1,p}(\Omega)}\leqslant\|(-\Delta+1)^\theta w\|_{L^q(\Omega)},\quad\text{provided that }1-\frac{n}{p}<2\theta-\frac{n}{q}. \tag{3.16}$$

Moreover, for $p<\infty$ the associated heat semigroup $(e^{t\Delta})_{t\geqslant0}$ maps $L^p(\Omega)$ into $D((-\Delta+1)^\theta)$ in any of the spaces $L^q(\Omega)$ for $q\geqslant p$, and there exist $c>0$ and $\gamma>0$ such that the $L^p-L^q$ estimate

$$\|(-\Delta+1)^\theta e^{t(\Delta-1)}w\|_{L^q(\Omega)}\leqslant ct^{-\theta-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}e^{-\gamma t}\|w\|_{L^p(\Omega)}\quad\text{for all }w\in L^p(\Omega) \tag{3.17}$$

holds.

There are two well-established methods to establish a uniform bound on $u$ in the literature: One way to derive such $L^\infty$ bounds is based on the iterative technique of Moser and Alikakos (cf. [2] or [6], for instance); the other uses semigroup arguments (cf. [14] or [28], for instance). Both techniques are standard; however, for completeness, we present a short proof here by a combination of the above two arguments. We first use the semigroup argument to establish a uniform bound on $\nabla v$, then employ the iterative technique to derive a uniform bound on $u$.

**Lemma 3.2.** *Let the assumptions in Theorem* 1.1 *hold. Then there exists a constant $C > 0$ such that the first component of solution of* (1.1) *satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C \quad \textit{for all } t \in (0, T_{max}). \tag{3.18}$$

**Proof.** In view of Lemma 2.1 it is sufficient to prove that for any $\tau \in (0, T_{max})$,

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leqslant C(\tau) \quad \text{for all } t \in (\tau, T_{max}) \tag{3.19}$$

holds with some $c(\tau) > 0$. To this end, we fix $\tau \in (0, T_{max})$ such that $\tau < 1$, pick $\theta \in (\frac{2n+1}{2n+2}, 1)$, and let $q := n+1$. Then from the representation formula

$$v(\cdot,t) = e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-s)(\Delta-1)}\big(1-u(\cdot,s)\big)v(\cdot,s)\,ds, \quad t \in (0, T_{max}),$$

we obtain in view of (2.1), (3.16), (3.17) and Lemma 3.1

$$\begin{aligned}
\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} &\leqslant c\big\|(-\Delta+1)^\theta v(\cdot,t)\big\|_{L^q(\Omega)} \\
&\leqslant ct^{-\theta}e^{-\gamma t}\|v_0\|_{L^q(\Omega)} + c\int_0^t (t-s)^{-\theta}e^{-\gamma(t-s)}\big(1+\|u(\cdot,s)\|_{L^q(\Omega)}\big)\|v(\cdot,s)\|_{L^\infty(\Omega)}\,ds \\
&\leqslant ct^{-\theta} + c\int_0^t (t-s)^{-\theta}e^{-\gamma(t-s)}\,ds \\
&\leqslant ct^{-\theta} + c\int_0^\infty \sigma^{-\theta}e^{-\gamma\sigma}\,d\sigma \\
&\leqslant c\big(\tau^{-\theta}+1\big) \quad \text{for all } t \in (\tau, T_{max}),
\end{aligned} \tag{3.20}$$

where $c$ denotes a generic constant that may vary from line to line.

For any $p \geqslant 2$, we derive from the first equation in (1.1), (3.20) and Young's inequality that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u^p &= p\int_\Omega u^{p-1}u_t \\
&= p\int_\Omega u^{p-1}\big(\Delta u - \nabla\cdot(\chi u\nabla v)\big) \\
&= -p(p-1)\int_\Omega u^{p-2}|\nabla u|^2 + \chi p(p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla v \\
&\leqslant -\frac{4(p-1)}{p}\int_\Omega \big|\nabla u^{\frac{p}{2}}\big|^2 + c_1p(p-1)\cdot\frac{2}{p}\int_\Omega u^{\frac{p}{2}}\cdot\big|\nabla u^{\frac{p}{2}}\big| \\
&\leqslant -\frac{4(p-1)}{p}\int_\Omega \big|\nabla u^{\frac{p}{2}}\big|^2 + c_1(p-1)\bigg(\frac{2}{c_1p}\int_\Omega \big|\nabla u^{\frac{p}{2}}\big|^2 + \frac{c_1p}{2}\int_\Omega u^p\bigg) \\
&= -\frac{2(p-1)}{p}\int_\Omega \big|\nabla u^{\frac{p}{2}}\big|^2 + \frac{c_1^2}{2}p(p-1)\int_\Omega u^p
\end{aligned}$$

where $c_1$ and in what follows the constants $c_i$ $(i \geqslant 2)$ are constants that are independent of $p$. Hence,

$$\frac{d}{dt}\int_\Omega u^p + p(p-1)\int_\Omega u^p \leqslant -\frac{2(p-1)}{p}\int_\Omega \big|\nabla u^{\frac{p}{2}}\big|^2 + c_2p(p-1)\int_\Omega u^p \quad \text{for all } p \geqslant 2, \tag{3.21}$$

where $c_2 := 1 + c_1^2/2$. Next we demonstrate that the last term on the right-hand side of (3.21) can be dominated by $\varepsilon\int_\Omega |\nabla u^{\frac{p}{2}}|^2$ and $(\int_\Omega u^{\frac{p}{2}})^2$ for some small $\varepsilon > 0$. To this end, we need the following interpolation inequality [17, p. 63]: For any $w \in W^{1,2}(\Omega)$,

$$\|w-\bar{w}\|_{L^2(\Omega)}^2 \leqslant c_3\|\nabla w\|_{L^2(\Omega)}^{2\eta}\|w\|_{L^1(\Omega)}^{2(1-\eta)},$$

where $\bar{w}=\frac{1}{|\Omega|}\int_\Omega w$, $\eta=n/(n+1)$, and $c_3$ is a constant depending only on $n$ and $\Omega$. This, along with Young's inequality ($yz\leqslant \varepsilon y^p + c\varepsilon^{-\frac{q}{p}}z^q$, $y,z>0$, $p,q>0$, $\frac{1}{p}+\frac{1}{q}=1$), entails that

$$\|w\|_{L^2(\Omega)}^2 \leqslant \varepsilon\|\nabla w\|_{L^2(\Omega)}^2 + c_4\big(1+\varepsilon^{-\frac{n}{2}}\big)\|w\|_{L^1(\Omega)}^2 \quad \text{for any } \varepsilon>0, \tag{3.22}$$

where $c_4>0$ depends only on $n$ and $\Omega$, but it is independent of $\varepsilon$. Applying interpolation inequality (3.22) with $w=u^{\frac{p}{2}}$ and $\varepsilon=\frac{2}{p^2c_2}$, we obtain

$$c_2p(p-1)\int_\Omega u^p \leqslant \frac{2(p-1)}{p}\int_\Omega \big|\nabla u^{\frac{p}{2}}\big|^2 + c_5p(p-1)\big(1+p^n\big)\bigg(\int_\Omega u^{\frac{p}{2}}\bigg)^2, \tag{3.23}$$

where $c_5:=c_2\max\{1,(\frac{c_2}{2})^{\frac{n}{2}}\}$. Inserting (3.23) into (3.21) and noting $1+p^n\leqslant(1+p)^n$, we obtain

$$\frac{d}{dt}\int_\Omega u^p + p(p-1)\int_\Omega u^p \leqslant c_5p(p-1)(1+p)^n\bigg(\int_\Omega u^{\frac{p}{2}}\bigg)^2. \tag{3.24}$$

Hence,

$$\frac{d}{dt}\bigg[e^{p(p-1)t}\int_\Omega u^p\bigg] \leqslant c_5e^{p(p-1)t}p(p-1)(1+p)^n\bigg(\int_\Omega u^{\frac{p}{2}}\bigg)^2. \tag{3.25}$$

Integrating (3.25) over the time interval $[0,t]$ for $0<t<T_{max}$, we obtain

$$\int_\Omega u^p(x,t) \leqslant \int_\Omega u_0^p(x) + c_5(1+p)^n \sup_{0\leqslant t\leqslant T_{max}}\bigg(\int_\Omega u^{\frac{p}{2}}(x,t)\bigg)^2. \tag{3.26}$$

Denote

$$K(p):=\max\bigg\{\|u_0\|_{L^\infty(\Omega)}, \sup_{0\leqslant t\leqslant T_{max}}\bigg(\int_\Omega u^p(x,t)\bigg)^{\frac{1}{p}}\bigg\}.$$

Then (3.26) yields

$$K(p)\leqslant\big[c_6(1+p)^n\big]^{\frac{1}{p}}K(p/2) \quad \text{for all } p\geqslant 2, \tag{3.27}$$

where $c_6:=|\Omega|+c_5$. Taking $p=2^j$, $j=1,2,\ldots$, one obtains

$$\begin{aligned}
K\big(2^j\big) &\leqslant c_6^{2^{-j}}\big(1+2^j\big)^{2^{-j}n}K\big(2^{j-1}\big)\\
&\leqslant \cdots\\
&\leqslant c_6^{2^{-j}+\cdots+2^{-1}}\big(1+2^j\big)^{2^{-j}n}\cdots(1+2)^{2^{-1}n}K(1)\\
&\leqslant c_6\big[2^{j2^{-j}n}\big(2^{-j}+1\big)^{2^{-j}n}\big]\cdots\big[2^{2^{-1}n}\big(2^{-1}+1\big)^{2^{-1}n}\big]K(1)\\
&\leqslant c_6 2^{[j2^{-j}+(j-1)2^{-(j-1)}+\cdots+2^{-1}]n}\cdot 2^{(2^{-j}+2^{-(j-1)}+\cdots+2^{-1})n}K(1)\\
&\leqslant c_6 2^{3n}K(1).
\end{aligned}$$

Letting $j\to\infty$ and using (2.3), we finally conclude that

$$\big\|u(\cdot,t)\big\|_{L^\infty(\Omega)} \leqslant c_6 2^{3n}K(1) \leqslant c_6 2^{3n}\max\big\{\|u_0\|_{L^\infty(\Omega)}, \|u_0\|_{L^1(\Omega)}\big\} \leqslant c.$$

This completes the proof of (3.18). □

We are now in the position to prove our main result.

**Proof of Theorem 1.1.** The assertion is an immediate consequence of Lemma 3.2, (2.1) and the extensibility criterion provided by Lemma 2.1. □

Finally, we remark that in order to extend the local solution to all $t > 0$ in the framework of Schauder theory, one needs to establish the following higher-order estimates:

$$\|(u, v)\|_{C^{2+\gamma_0,(2+\gamma_0)/2}(\Omega\times[0,T])} \leqslant C(T)$$

for any given $T > 0$ (cf. [4,24], for instance). However, in the framework of Amann's theory, to achieve the global existence, one needs only to establish estimate (2.2).

## Acknowledgment

The author is grateful to an anonymous referee for valuable comments.

## References


[1] J. Adler, Chemotaxis in bacteria, Science 153 (1966) 708–716.
[2] N.D. Alikakos, $L^p$ bounds of solutions of reaction–diffusion equations, Comm. Partial Differential Equations 4 (1979) 827–868.
[3] H. Amann, Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems, in: H.J. Schmeisser, H. Triebel (Eds.), Function Spaces, Differential Operators and Nonlinear Analysis, in: Teubner-Texte Math., vol. 133, Teubner, Stuttgart, Leipzig, 1993, pp. 9–126.
[4] Y.S. Choi, Z.A. Wang, Prevention of blow-up by fast diffusion in chemotaxis, J. Math. Anal. Appl. 362 (2010) 553–564.
[5] T. Cieślak, Quasilinear nonuniformly parabolic system modelling chemotaxis, J. Math. Anal. Appl. 326 (2007) 1410–1426.
[6] L. Corrias, B. Perthame, H. Zaag, $L^p$ and $L^\infty$ a priori estimates for some chemotaxis models and applications to the Cauchy problem (dynamics of spatio–temporal patterns for the system of reaction–diffusion equations), Kyoto Univ., RIMS 1416 (2005) 105–119.
[7] M. Delgado, I. Gayte, C. Morales-Rodrigo, A. Suárez, An angiogenesis model with nonlinear chemotactic response and flux at the tumor boundary, Nonlinear Anal. 72 (2010) 330–347.
[8] M. Di Francesco, A. Lorz, P. Markowich, Chemotaxis–fluid coupled model for swimming bacteria with nonlinear diffusion: global existence and asymptotic behavior, Discrete Contin. Dyn. Syst. Ser. A 28 (2010) 1437–1453.
[9] R.J. Duan, A. Lorz, P. Markowich, Global solutions to the coupled chemotaxis–fluid equations, Comm. Partial Differential Equations 35 (2010) 1635–1673.
[10] A. Friedman, Partial Differential Equations, Holt, Rinehart & Winston, New York, 1969.
[11] M.A. Herrero, J.L.L. Velázquez, A blow-up mechanism for a chemotaxis model, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 24 (1997) 633–683.
[12] T. Hillen, K.J. Painter, A user's guide to PDE models for chemotaxis, J. Math. Biol. 58 (2009) 183–217.
[13] D. Horstmann, G. Wang, Blow-up in a chemotaxis model without symmetry assumptions, European J. Appl. Math. 12 (2001) 159–177.
[14] D. Horstmann, M. Winkler, Boundedness vs. blow-up in a chemotaxis system, J. Differential Equations 215 (2005) 52–107.
[15] E.F. Keller, L.A. Segel, Traveling bands of chemotactic bacteria: a theoretical analysis, J. Theoret. Biol. 30 (1971) 377–380.
[16] E.F. Keller, L.A. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theoret. Biol. 26 (1970) 399–415.
[17] O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva, Linear and Quasi-Linear Equations of Parabolic Type, Amer. Math. Soc. Transl., vol. 23, Amer. Math. Soc., Providence, RI, 1968.
[18] T. Li, Z.A. Wang, Asymptotic nonlinear stability of traveling waves to conservation laws arising from chemotaxis, J. Differential Equations 250 (2011) 1310–1333.
[19] A. Lorz, Coupled chemotaxis fluid model, Math. Models Methods Appl. Sci. 20 (2010) 987–1004.
[20] T. Nagai, T. Ikeda, Traveling waves in a chemotaxis model, J. Math. Biol. 30 (1991) 169–184.
[21] T. Nagai, T. Senba, K. Yoshida, Application of the Trudinger–Moser inequality to a parabolic system of chemotaxis, Funkcial. Ekvac. Ser. Int. 40 (1997) 411–433.
[22] L. Nirenberg, An extended interpolation inequality, Ann. Sc. Norm. Super. Pisa Cl. Sci. (3) 20 (1966) 733–737.
[23] K. Osaki, A. Yagi, Finite dimensional attractors for one-dimensional Keller–Segel equations, Funkcial. Ekvac. 44 (2001) 441–469.
[24] Y. Tao, C. Cui, A density-dependent chemotaxis–haptotaxis system modeling cancer invasion, J. Math. Anal. Appl. 367 (2010) 612–624.
[25] Y. Tao, M. Winkler, Global existence and boundedness in a Keller–Segel–Stokes model with arbitrary porous medium diffusion, preprint.
[26] I. Tuval, L. Cisnerous, C. Dombrowski, C.W. Wolgemuth, J.O. Kessler, R.E. Goldstein, Bacterial swimming and oxygen transport near contact lines, Proc. Natl. Acad. Sci. USA 102 (2005) 2277–2282.
[27] Z.A. Wang, T. Hillen, Shock formation in a chemotaxis model, Math. Methods Appl. Sci. 31 (2008) 45–70.
[28] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller–Segel model, J. Differential Equations 248 (2010) 2889–2905.
[29] M. Winkler, Absence of collapse in a parabolic chemotaxis system with signal-dependent sensitivity, Math. Nachr. 283 (2010) 1664–1673.
[30] A. Yagi, Norm behavior of solutions to a parabolic system of chemotaxis, Math. Jpn. 45 (1997) 241–265.